# Incentive Design and Learning in Large Populations: A System-Theoretic Approach With Applications To Epidemic Mitigation

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December 15, 2024

A. JAMES CLARK SCHOOL OF ENGINEERING

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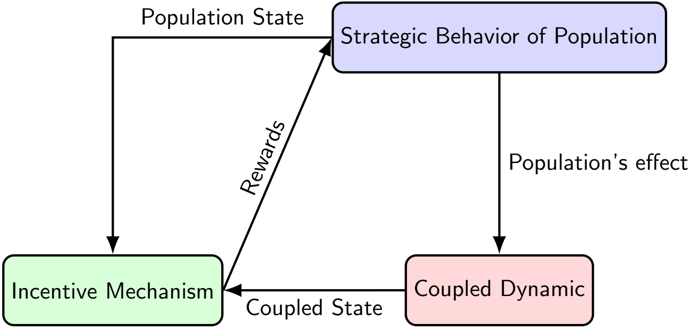
# "Big Picture"


Population State
Strategic Behavior of Population
Rewards
Population's effect
Incentive Mechanism
Coupled State
Coupled Dynamic

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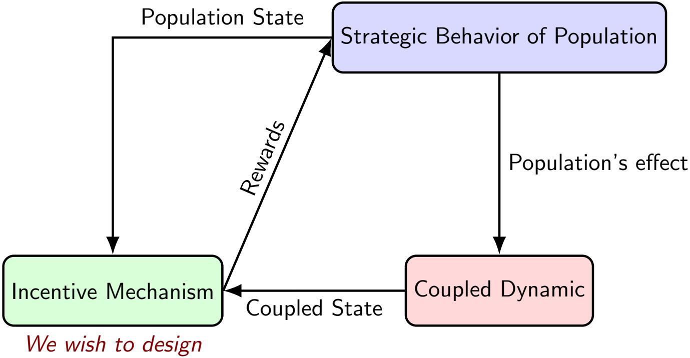
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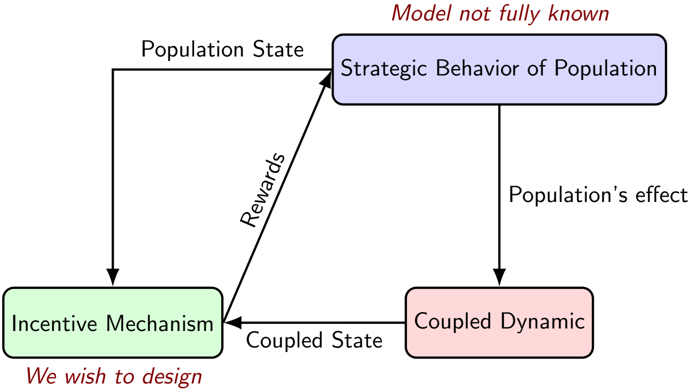
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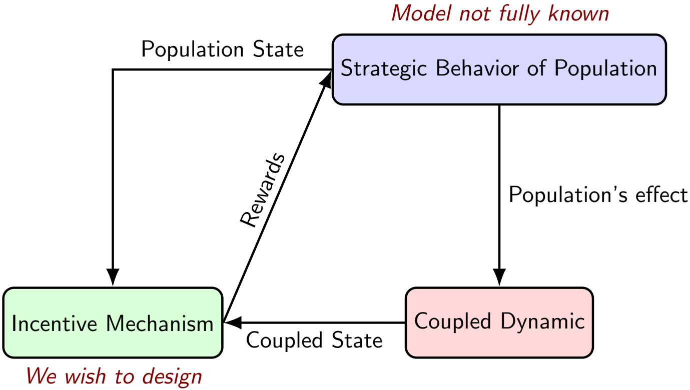


**GOAL: Incentivize Population To Stabilize Desirable Equilibria**

# Epidemic Mitigation

# Epidemics Research in the Control Systems Community

*Partial list of related articles:*


- *E. D. Sontag, "An explicit formula for minimizing the infected peak in an SIR epidemic model when using a fixed number of complete lockdowns," International Journal of Robust and Nonlinear Control (2021), pp. 1-24.*
- *P.E. Paré, C.L. Beck, T. Basar, "Modeling, estimation, and analysis of epidemics over networks: an overview," Annual Reviews in Control, 50 (2020), pp. 345-360.*
- *W. Mei, S. Mohagheghi, S. Zampieri, F. Bullo, "On the dynamics of deterministic epidemic propagation over networks Annual Reviews in Control," 44 (2017), pp. 116-128.*
- *C. Nowzari, V. M. Preciado, and G. J. Pappas, "Optimal resource allocation for control of networked epidemic models," IEEE TCNS, vol. 4, no. 2, pp. 159–169, Jun. 2017.*
- *V. M. Preciado, M. Zargham, C. Enyioha, A. Jadbabaie, and G. J. Pappas, "Optimal resource allocation for network protection against spreading processes," IEEE TCNS, vol. 1, no. 1, pp. 99– 108, Mar. 2014.*

# Incorporating A Population's Strategic Behavior

- A. Ahuja, "How a US agency hopes to predict disease just like the weather," Financial Times, February 2021.
  - *Modeling and mitigation of epidemics are national priorities.*
  - *"Epidemiology is not physics. A large part of epidemiological modelling is based on human behaviour, which remains largely unpredictable", Professor Graham Medley head of infectious disease modelling at the London School of Hygiene and Tropical Medicine.*

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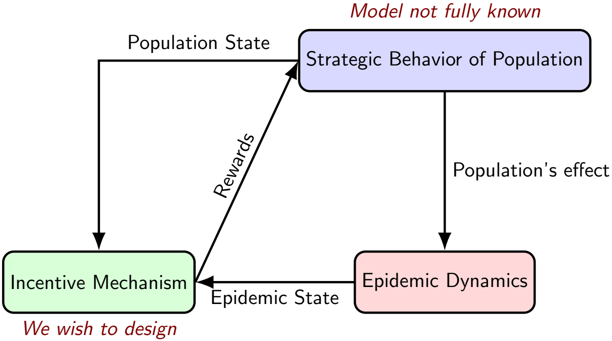
- J. Interlandi, "Inside the C.D.C.'s Pandemic 'Weather Service'," The New York Times, November 2021.
  - *"We know that people's behavior, the mode of transmission and the virus's characteristics all play a role. But we don't have a detailed, quantitative understanding of how all these forces interact." With Covid, the biggest wild card has been human behavior.*

# Epidemics Research and Social Dynamics

*Partial list of related articles:*


- *K Paarporn, C. Eksin, "SIS epidemics coupled with evolutionary social distancing dynamics," 2023 American Control Conference (ACC), 2023.*
- *B. Buonomo, P. Manfredi, and A. dOnofrio, "Optimal time-profiles of public health intervention to shape voluntary vaccination for childhood diseases," Mathematical Biology, vol. 78, pp. 1089–1123, Mar. 2019.*
- *A. R. Hota and S. Sundaram, "Game-theoretic vaccination against networked SIS epidemics and impacts of human decision-making," IEEE Control Netw. Syst., vol. 6, no. 4, pp. 1461–1472, Dec. 2019.*
- *M.A.Amaral, M.M.deOliveira, and M.A.Javarone,"An epidemiological model with voluntary quarantine strategies governed by evolutionary game dynamics," Chaos, Solitons & Fractals, vol. 143, p. 110616, Feb. 2021.*
- *K. A. Kabir and J. Tanimoto, "Evolutionary game theory modeling to represent the behavioural dynamics of economic shutdowns and shield immunity in the COVID-19 pandemic," R. Soc. Open Sci., vol. 7, no. 201095, Sep. 2020.*

# Overarching Problem Formulation



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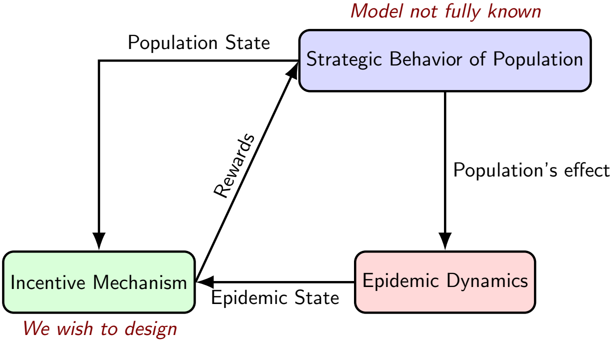


**GOAL: Nudge Population Towards Optimal Endemic Equilibrium**

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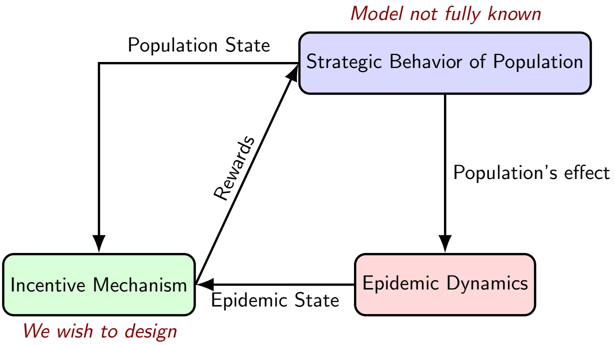


**GOAL: Nudge Population Towards Optimal Endemic Equilibrium**

- *Systematic design method, with guarantees.*
- *Optimal endemic equilibrium: lowest fraction of infected individuals.*
- *Budgetary constraints on long term average spending.*

# Technical Framework

# Key References For Our Approach and Framework

All the material covered in this talk is discussed in detail in the papers:

- *N.C. Martins, J. Certório, R.J. La, " Epidemic population games and evolutionary dynamics," Automatica, 2023.*
- *J. Certório, N.C. Martins, R.J. La, "Epidemic Population Games With Nonnegligible Disease Death Rate," IEEE Control Systems Letters 6, 3229-3234.*

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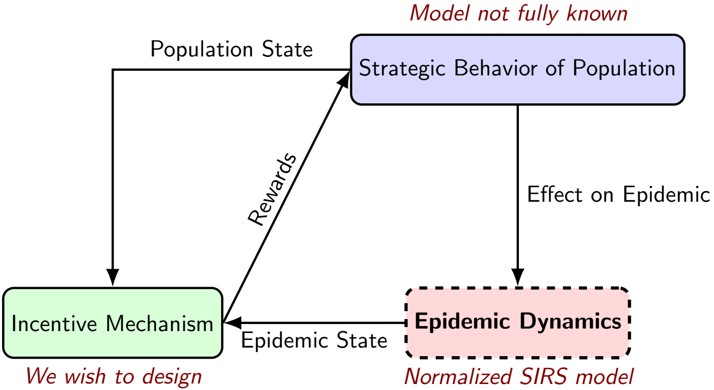
Followup work:


- *J. Certório, R.J. La, N.C. Martins, "Epidemic Population Games for Policy Design: Two Populations with Viral Reservoir Case Study," CDC, 2023.*
- *J. Certório, N.C. Martins, R. J. La and M. Arcak, "Incentive Designs for Learning Agents to Stabilize Coupled Exogenous Systems (I)," CDC, 2024.*

# Normalized SIRS Model



# Normalized SIRS Model

## Definition (Susceptible-Infected-Recovered-Susceptible)

*Normalizing* **(Kermack & McKendrick'27)**, *we get*

$$\dot{I}(t) = \mathcal{B}(t)\big(S(t)I(t)\big) - \sigma I(t),$$
$$\dot{R}(t) = \gamma I(t) - \omega R(t),$$

*where $I$, $R$ and $S := (1 - I - R)$ take values in $[0, 1]$ and are the proportions of the population which are infectious, have recovered and are susceptible to infection, respectively. Here $\sigma := \gamma + \theta$ and $\omega := \psi + \theta$, where $\gamma$ and $\psi$ denote the daily recovery rate and the daily rate at which recovered individuals become susceptible (due to waning immunity), respectively. The daily birth rate is $\theta$ (newborns are susceptible).*

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The estimated mean recovery time and immunity duration for COVID-19 are respectively approximately 10 days and 2-9 months, yielding $\gamma \approx 0.1$ and $\psi \in [0.0037, 0.017]$. Recall: $\sigma := \gamma + \theta$ and $\omega := \psi + \theta$.

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## Remark

*Notice that for constant* $\mathcal{B} = \beta > \sigma$, *the non-trivial endemic equilibrium is*

$$I^e := \eta(1 - \tfrac{\sigma}{\beta}),\ R^e := (1-\eta)(1 - \tfrac{\sigma}{\beta}), \quad \eta := \tfrac{\omega}{\omega+\gamma}.$$

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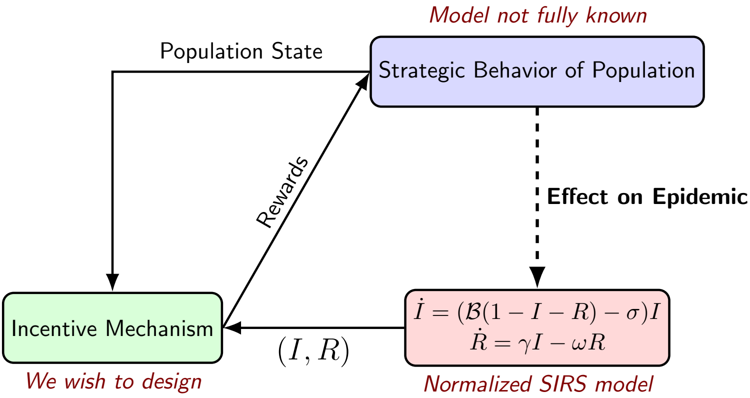
*With $\mathcal{B} = \beta > \sigma$, we get the Lyapunov function (O'Regan et al.'10)*

$$\mathfrak{L}(I, R) = (I - I^e) + I^e \ln \frac{I^e}{I} + \frac{\beta}{2\gamma}(R - R^e)^2, \ (I, R) \in (0, 1] \times [0, 1].$$

# Transmission Rate Model



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*Strategy Examples*

1  2  3 

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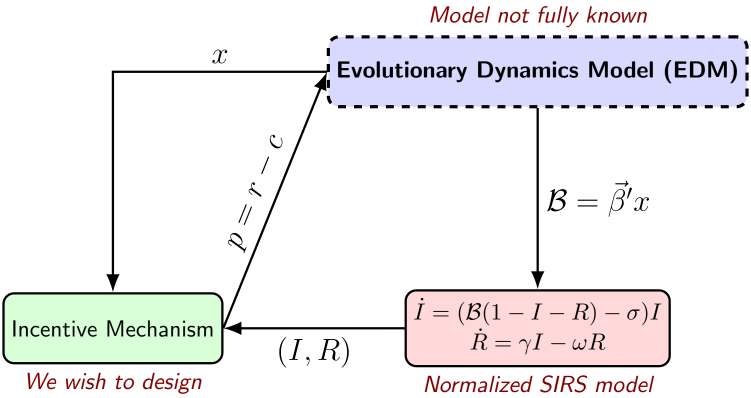
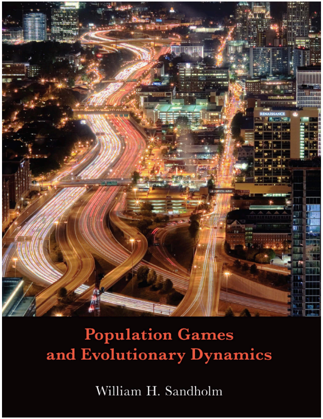
# Strategic Behavior of Population



# Evolutionary dynamics model (EDM)

Population game approach:

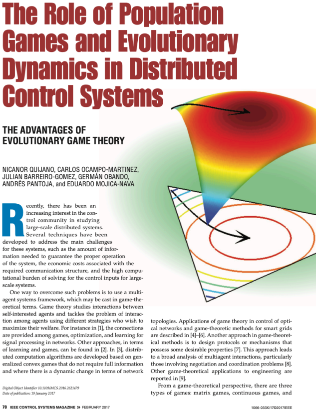
- W. H. Sandholm, *"Population Games and Evolutionary Dynamics,"* MIT Press, 2010.
- W. H. Sandholm, *"Population Games and Deterministic Evolutionary Dynamics,"* Handbook of Game Theory (Young & Zamir), North Holland, 2015.



# Evolutionary dynamics model (EDM)

Applications to control systems:

- N. Quijano, C. Ocampo-Martinez, J. Barreiro-Gomez, G. Obando, A. Pantoja, E. Mojica-Nava, *"The Role of Population Games and Evolutionary Dynamics in Distributed Control Systems,"* IEEE Control Systems Magazine, Feb. 2017.



# The Role of Population Games and Evolutionary Dynamics in Distributed Control Systems

THE ADVANTAGES OF EVOLUTIONARY GAME THEORY

NICANOR QUIJANO, CARLOS OCAMPO-MARTINEZ, JULIAN BARREIRO-GOMEZ, GERMÁN OBANDO, ANDRÉS PANTOJA, and EDUARDO MOJICA-NAVA

Recently, there has been an increasing interest in the control community in studying large-scale distributed systems. Several techniques have been developed to address the main challenges for these systems, such as the amount of information needed to guarantee the proper operation of the system, the economic costs associated with the required communication structure, and the high computational burden of solving for the control inputs for large-scale systems.

One way to overcome such problems is to use a multiagent systems framework, which may be cast in game-theoretical terms. Game theory studies interactions between self-interested agents and tackles the problem of interaction among agents using different strategies who wish to maximize their welfare. For instance in [1], the connections are provided among games, optimization, and learning for signal processing in networks. Other approaches, in terms of learning and games, can be found in [2]. In [3], distributed computation algorithms are developed based on generalized convex games that do not require full information and where there is a dynamic change in terms of network topologies. Applications of game theory in control of optical networks and game-theoretic methods for smart grids are described in [4]–[6]. Another approach in game-theoretical methods is to design protocols or mechanisms that possess some desirable properties [7]. This approach leads to a broad analysis of multiagent interactions, particularly those involving negotiation and coordination problems [8]. Other game-theoretical applications to engineering are reported in [9].

From a game-theoretical perspective, there are three types of games: matrix games, continuous games, and

*Digital Object Identifier 10.1109/MCS.2016.2621479*
*Date of publication: 19 January 2017*



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$$\dot{x}_i = \mathcal{V}_i(x,p) := \underbrace{\sum_{j=1}^{n} x_j \mathcal{T}_{ji}(x,p)}_{\text{switching rate towards } i} - \underbrace{\sum_{j=1}^{n} x_i \mathcal{T}_{ij}(x,p)}_{\text{switching rate out of } i}, \; 1 \leq i \leq n$$

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- $\mathcal{T}_{ij} : \mathbb{X} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is Lipschitz continuous, $i$ and $j$ in $\{1, \ldots, n\}$.
  - Learning rule (revision protocol), models strategic preferences.
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- There is continuous $\mathcal{P} : \mathbb{X} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and continuously differentiable $\mathcal{S} : \mathbb{X} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that:

$$\nabla_x \mathcal{S}(x,p)' \mathcal{V}(x,p) \leq -\mathcal{P}(x,p).$$
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Along trajectories, we get

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- $\mathcal{S}(x,p) = 0 \Leftrightarrow \mathcal{P}(x,p) = 0 \Leftrightarrow (x'p = \max_{1 \leq i \leq n} p_i) \Leftrightarrow \mathcal{V}(x,p) = 0.$

# Evolutionary dynamics model (EDM)

$$\dot{x}_i = \mathcal{V}_i(x,p) := \sum_{j=1}^n x_j \mathcal{T}_{ji}(x,p) - \sum_{j=1}^n x_i \mathcal{T}_{ij}(x,p), \ 1 \leq i \leq n$$

(Assumption) The following $\delta$-passivity conditions hold:

- There is continuous $\mathcal{P} : \mathbb{X} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and continuously differentiable $\mathcal{S} : \mathbb{X} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that:

  $\nabla_x \mathcal{S}(x,p)' \mathcal{V}(x,p) \leq -\mathcal{P}(x,p)$.
  $\nabla_p \mathcal{S}(x,p) = \mathcal{V}(x,p)$.

  Along trajectories, we get

  $$\frac{d}{dt}\mathcal{S}(x(t),p(t)) \leq -\mathcal{P}(x(t),p(t)) + \dot{x}(t)'\dot{p}(t).$$

- $\mathcal{S}(x,p) = 0 \Leftrightarrow \mathcal{P}(x,p) = 0 \Leftrightarrow (x'p = \max_{1 \leq i \leq n} p_i) \Leftrightarrow \mathcal{V}(x,p) = 0$.
- (non standard) $\mathcal{P}(x, \alpha p) \geq \mathcal{P}(x,p), \ \alpha > 1, \ x \in \mathbb{X}, \ p \in \mathbb{R}^n$.

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- There are continuous nondecreasing $\phi_j$ such that
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- (Smith'84) used $\mathcal{T}_{ij}^{\text{Smith}}(x,p) = [p_j - p_i]_+$.
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# Key References For $\delta$-passivity


- M. J. Fox and J. S. Shamma, "Population games, stable games, and passivity," Games, 2013.
- S. Park, N. C. Martins and J. S. Shamma," From Population Games to Payoff Dynamics Models: A Passivity-Based Approach", IEEE CDC, 2019. (tutorial paper)
- M. Arcak and N. C. Martins, "Dissipativity tools for convergence to Nash equilibria in population games," IEEE TCNS, 2020.
- S. Kara and N. C. Martins,"Pairwise Comparison Evolutionary Dynamics with Strategy-Dependent Revision Rates: Stability and $\delta$-Passivity," IEEE TCNS, 2020.
- See also the following article drastically expanding the class of $\delta$-passive rules to include the so-called hybrid rules combining elements of established classes and, for the first time, also best response behaviors:
  - Certório, Chang, Martins, Nuzzo, Shoukry, "Passivity Tools for Hybrid Learning Rules in Large Populations," arXiv:2407.02083

# Other related passivity results (partial list)

- B. Gao and L. Pavel, "On Passivity, Reinforcement Learning, and Higher Order Learning in Multiagent Finite Games," IEEE TAC, 2020.
- M. A. Mabrok, "Passivity Analysis of Replicator Dynamics and Its Variations," IEEE TAC, 2021.
- Nuno C. Martins, Jair Certório, Matthew S Hankins, "Counterclockwise Dissipativity, Potential Games and Evolutionary Nash Equilibrium Learning," arXiv:2408.00647. (A lot to explore!)

# Ovearching Approach: $\delta$-passivity Based Design

This talk can be viewed as a case-study on the following advantages of $\delta$-passivity for design:

- It is equilibrium independent (Hines, Arcak, Packard'11).
  - The selection of a desirable (optimal) equilibrium can be carried out separately from its stabilization.

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- Storage functions can be constructed from individual components' Lyapunov functions (or modifications).
- Storage functions of the individual components can be used to construct a Lyapunov function for the overall system.
  - May lead to useful bounds.
  - Is important for the theory of ODE approximation.

# Optimal Equilibrium

# Assumptions On Transmission Rate Coupling

## Assumption

*The strategies' inherent costs decrease for higher transmission rates, and we order the entries of $\vec{\beta}$ and $c$ as:*

$$\vec{\beta}_i < \vec{\beta}_{i+1} \text{ and } c_i > c_{i+1}, \quad 1 \leq i \leq n-1.$$

*We consider that $\vec{\beta}_1 > \sigma$, i.e., a transmission rate less than or equal to $\sigma$ would be unfeasible or too onerous.*

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*The following must hold when $n \geq 3$:*

$$\frac{c_i - c_{i+1}}{\vec{\beta}_{i+1} - \vec{\beta}_i} > \frac{c_{i+1} - c_{i+2}}{\vec{\beta}_{i+2} - \vec{\beta}_{i+1}}, \quad 1 \leq i \leq n-2.$$

# Optimal endemic transmission rate

We will use $\tilde{c}$ defined below to specify cost constraints because for a planner seeking to promote the $i$-th strategy it suffices to offer incentives to offset the differential $\tilde{c}_i$.

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## Definition

*Given a cost budget $c^*$ in $(0, \tilde{c}_1)$ with $c^* \notin \cup_{i=1}^n \{\tilde{c}_i\}$, we determine the optimal endemic transmission rate $\beta^*$ as:*

$$\beta^* := \min \left\{ \vec{\beta}' x \mid \tilde{c}' x \leq c^*, \; x \in \mathbb{X} \right\}.$$

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## Proposition

*It follows from KKT that the unique solution is*

$$x^* := \arg\min \left\{ \vec{\beta}' x \mid \tilde{c}' x \leq c^*,\ x \in \mathbb{X} \right\}.$$

*With $\tilde{c}_{i^*+1} < c^* < \tilde{c}_{i^*}$, it results that $x^*_{i^*} = (c^* - \tilde{c}_{i^*+1})/(\tilde{c}_{i^*} - \tilde{c}_{i^*+1})$, $x^*_{i^*+1} = 1 - x^*_{i^*}$ and the other entries of $x^*$ are zero.*

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## Definition

*From* $\mathcal{B} = \beta^*$*, the optimal SIR endemic equilibrium (*$I^*$ *is minimized) is*

$$I^* := \eta(1 - \tfrac{\sigma}{\beta^*}),\ R^* := (1-\eta)(1 - \tfrac{\sigma}{\beta^*}), \quad \eta := \tfrac{\omega}{\omega+\gamma}.$$

# EPG

# Epidemic Population Game

## Definition (EPG)

*Below, we specify an epidemic population game (EPG):*

$$\dot{I}(t) = \big(\mathcal{B}(t)(1 - I(t) - R(t)) - \sigma\big)I(t),$$
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\dot{q}(t) &= G(I(t), R(t), x(t), q(t)), \\
r(t) &= H(I(t), R(t), x(t), q(t)),
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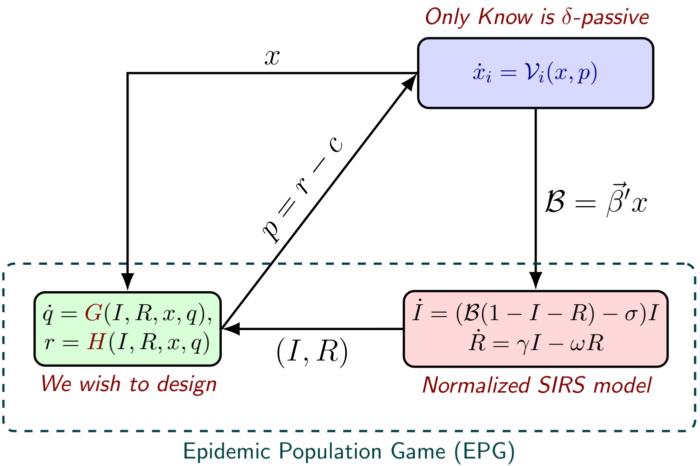
*with* $q(t)$ *in* $\mathbb{R}^m$ *for some* $m$.

# Epidemic Population Game



# Stabilization: Problem Formulation

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## Problem Statement

*Design $G$ and $H$ such that $(I^*, R^*, x^*, q^*)$, with $q^* := 0$, is the unique globally asymptotically stable equilibrium point for any $(I(0), R(0), x(0), q(0))$ in $(0, 1] \times [0, 1] \times \mathbb{X} \times \mathbb{R}^m$.*

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  - Can use the Lyapunov function to obtain bounds for $I$.

# A Stabilizing Solution

# A Solution For Any $\delta$-passive (EDM)

**Definition (Reference epidemic variables)**

$$\hat{I}(t) := \eta\left(1 - \frac{\sigma}{\mathcal{B}(t)}\right), \;\; \hat{R}(t) := (1-\eta)\left(1 - \frac{\sigma}{\mathcal{B}(t)}\right).$$

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The following is a solution (note: omit $(t)$ for simplicity):

$$\begin{aligned}\dot{q} = G(I,R,x,q) &:= (\hat{I} - I) + \eta(\ln I - \ln \hat{I}) + \upsilon^2(\beta^* - \mathcal{B}) \\ &\quad + \tfrac{\mathcal{B}}{\gamma}(R - \hat{R})(1 - \eta - R), \\ r = H(I,R,x,q) &:= q\vec{\beta} + r^*, \quad p = r - c\end{aligned}$$

where $q \in \mathbb{R}$. Here $\upsilon > 0$ and $\rho^* > 0$ are design parameters, and

$$r_i^* := \begin{cases} \tilde{c}_i - \rho^* & \text{if } x_i^* = 0 \\ \tilde{c}_i & \text{otherwise} \end{cases}, \quad 1 \leq i \leq n.$$

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The following is an approximation (note: omit $(t)$ for simplicity):

$$\dot{q} = G(I, R, x, q) \underset{R,\, I \ll 1}{\approx} \eta(\ln I - \ln \hat{I}) + \upsilon^2(\beta^* - \mathcal{B}),$$

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At the desired equilibrium ($q^* = 0$):

$$p_i^* := \begin{cases} -\tilde{c}_n - \rho^* & \text{if } x_i^* = 0 \\ -\tilde{c}_n & \text{otherwise} \end{cases}, \quad 1 \leq i \leq n.$$

# Main Theorem

$$\dot{q} = G(I,R,x,q) := (\hat{I} - I) + \eta(\ln I - \ln \hat{I}) + \upsilon^2(\beta^* - \mathcal{B}) + \frac{\mathcal{B}}{\gamma}(R - \hat{R})(1 - \eta - R),$$

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## Theorem

*For any $\delta$-passive (EDM), the following holds for $G$ and $H$ (universally):*

- ⇨ $(I^*, R^*, x^*, 0)$ *is the unique equilibrium (assuming $I^*(0) > 0$).*
- ⇨ *The equilibrium is globally asymptotically stable, with $I(0) > 0$.*
  - ▶ *A global Lyapunov function exists.*

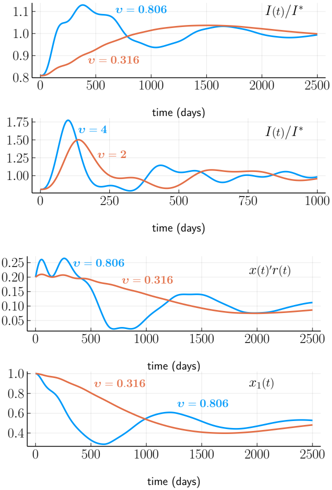
# Stabilization Policy in Action $(n = 2)$

We consider the learning rule

$$\mathcal{T}_{ij}(x,p) = 0.1 \min\{1, [p_j - p_i]_+\},$$

and parameters for COVID-19:

- $\sigma = 0.1$ (infectiousness period $\sim$ 10 days), $\gamma = \sigma$, and $\omega = 0.005$ (immunity period $\sim$ 200 days).
- $\vec{\beta}_1 = 0.15$, $\vec{\beta}_2 = 0.19$, while the cost vector is $c_1 = 0.2$, $c_2 = 0$. We select $c^* = 0.1$, which gives $\beta^* = 0.17$, $x_1^* = x_2^* = 0.5$, and $(I^*, R^*) \approx (1.96\%, 39.22\%)$.
- $x_1(0) = 1$, $(I(0), R(0)) = (\hat{I}(0), \hat{R}(0)) = (1.60\%, 31.75\%)$, and $\mathcal{B}(0) = 0.15$.



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$$\mathcal{L}(\mathscr{Y}) := \mathcal{S}(x,p) + \mathscr{S}(\mathcal{I},\mathcal{R},\mathcal{B}), \quad \mathscr{Y} \in \mathbb{Y},$$
$$\mathscr{Y} := (\mathcal{I},\mathcal{R},x,q), \quad \mathcal{B} := \vec{\beta}' x,$$

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- When computing the derivative of $\mathcal{L}(\mathscr{Y}(t))$ along trajectories and using $\delta$-passivity, the choice of $\dot{q}$ becomes clear.
- Best response map is used to show that $\mathcal{L}(\mathscr{Y})$ is zero iff $\mathscr{Y}$ is the desired equilibrium $(\mathcal{I}^*, \mathcal{R}^*, x^*, 0)$.

# Outlining the details

From $\delta$-passivity, the storage function satisfies

$$\begin{aligned}
\tfrac{d}{dt}\mathcal{S}(x(t),p(t)) &\leq -\mathcal{P}(x(t),p(t)) + \dot{p}(t)'\dot{x}(t) \\
&= -\mathcal{P}(x(t),p(t)) + (\dot{q}(t)\vec{\beta})'\dot{x}(t) \\
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Finally, when taking derivatives of $\mathscr{S}$ along trajectories we get

$$\tfrac{d}{dt}\mathscr{S}(\mathcal{I}(t),\mathcal{R}(t),\mathcal{B}(t)) = -\tilde{\mathcal{I}}(t)^2 - \tfrac{\omega}{\gamma}\tilde{\mathcal{R}}(t)^2 - G(\cdot)\dot{\mathcal{B}}(t),$$

$$G(\cdot) := \partial_{\mathcal{B}}\mathscr{S}(\cdot).$$

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$$\begin{aligned}\tfrac{d}{dt}\mathcal{S}(x(t),p(t)) &\leq -\mathcal{P}(x(t),p(t)) + \dot{p}(t)'\dot{x}(t) \\ &= -\mathcal{P}(x(t),p(t)) + (\dot{q}(t)\vec{\beta})'\dot{x}(t) \\ &= -\mathcal{P}(x(t),p(t)) + \dot{q}(t)\dot{\mathcal{B}}(t).\end{aligned}$$

Finally, when taking derivatives of $\mathscr{S}$ along trajectories we get

$$\tfrac{d}{dt}\mathscr{S}(\mathcal{I}(t),\mathcal{R}(t),\mathcal{B}(t)) = -\tilde{\mathcal{I}}(t)^2 - \tfrac{\omega}{\gamma}\tilde{\mathcal{R}}(t)^2 - G(\cdot)\dot{\mathcal{B}}(t),$$
$$G(\cdot) := \partial_{\mathcal{B}}\mathscr{S}(\cdot).$$

Choose $\dot{q} = G(\cdot)$. The derivative of $\mathcal{L}(\mathscr{Y}) := \mathcal{S}(x,p) + \mathscr{S}(\mathcal{I},\mathcal{R},\mathcal{B})$ along trajectories is

$$\tfrac{d}{dt}\mathcal{S}(x(t),p(t)) + \tfrac{d}{dt}\mathscr{S}(\mathcal{I}(t),\mathcal{R}(t),\mathcal{B}(t)) \leq -\mathcal{P}(x(t),p(t)) - \tilde{\mathcal{I}}(t)^2 - \tfrac{\omega}{\gamma}\tilde{\mathcal{R}}(t)^2.$$

# Details about $\mathscr{S}$ (modified Lyap. func. for SIRS model)

$$\mathscr{S}(\mathcal{I},\mathcal{R},\mathcal{B}) := \hat{\mathcal{I}}\ln\frac{\hat{\mathcal{I}}}{\mathcal{I}} - \tilde{\mathcal{I}} + \frac{1}{2\gamma}\tilde{\mathcal{R}}^2 + \frac{\upsilon^2}{2}(\mathcal{B}-\beta^*)^2.$$

Where we define:

$$\begin{aligned}
\mathcal{I}(t) &:= \mathcal{B}(t)I(t), & \mathcal{R}(t) &:= \mathcal{B}(t)R(t),\\
\hat{\mathcal{I}}(t) &:= \mathcal{B}(t)\hat{I}(t), & \hat{\mathcal{R}}(t) &:= \mathcal{B}(t)\hat{R}(t),\\
\tilde{\mathcal{I}}(t) &:= \hat{\mathcal{I}}(t) - \mathcal{I}(t), & \tilde{\mathcal{R}}(t) &:= \hat{\mathcal{R}}(t) - \mathcal{R}(t),
\end{aligned}$$

Recall also that:

$$\hat{I} := \eta\Big(1-\frac{\sigma}{\mathcal{B}}\Big),\ \hat{R} := (1-\eta)\Big(1-\frac{\sigma}{\mathcal{B}}\Big),$$

$$\mathcal{I}^* := \mathcal{B}^* I^*,\ I^* := \eta\Big(1-\frac{\sigma}{\mathcal{B}^*}\Big),$$

$$\mathcal{R}^* := \mathcal{B}^* R^*,\ R^* := (1-\eta)\Big(1-\frac{\sigma}{\mathcal{B}^*}\Big).$$

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Important properties of $\mathscr{S}$:

- $\mathscr{S}$ is convex.
- $\mathscr{S}$ is nonnegative and $\mathscr{S}(\mathcal{I},\mathcal{R},\mathcal{B}) = 0$ iff $(\mathcal{I},\mathcal{R},\mathcal{B}) = (\mathcal{I}^*,\mathcal{R}^*,\mathcal{B}^*)$.

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Now recall that

$$\begin{aligned} \mathcal{L}(\mathscr{Y}) :=& \mathcal{S}(x,p) + \mathscr{S}(\mathcal{I},\mathcal{R},\mathcal{B}), \quad \mathscr{Y} \in \mathbb{Y}, \\ \mathscr{Y} :=& (\mathcal{I},\mathcal{R},x,q), \quad \mathcal{B} := \vec{\beta}' x, \end{aligned}$$

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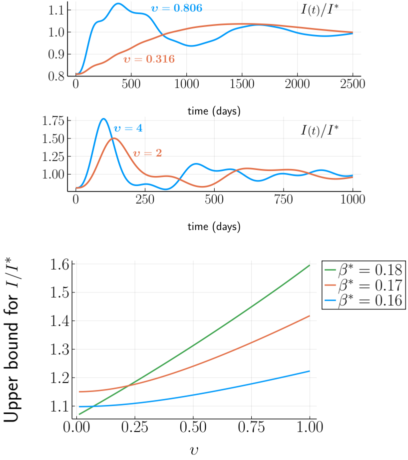
When $\mathcal{S}(x(0),p(0)) = 0$, we obtain the useful inequality

$$\mathscr{S}(\mathcal{I}(t),\mathcal{R}(t),\mathcal{B}(t)) \le \mathcal{L}(\mathscr{Y}(t)) \le \mathcal{L}(\mathscr{Y}(0)) = \mathscr{S}(\mathcal{I}(0),\mathcal{R}(0),\mathcal{B}(0)).$$

# Using the Upper Bound to Select $\upsilon$: An Example

- Initial condition is equilibrium for a budget of $c^* = 0.2$.
- New budget is relaxed to $c^* = 0.1$.
- Bound for $I/I^*$
  - is a function of $\upsilon$.
  - is universal, i.e., valid for any $\delta$-passive (EDM).
  - computed efficiently as a quasi-convex program.
- Goal of $I/I^* \leq 1.34$ gives $\upsilon \leq 0.806$.



# Thanks

# Future Directions and Thanks

The following are worthwhile future directions:

- Explore applications to other coupled dynamics.
- Are our bounds tight? If so, obtain better ones.
- Consider multiple populations (see Certorio et. all. CDC'23).
- Generalize beyond $\delta$-passive (EDM).

Thanks to our sources of support:

**AFOSR FA95502310467**,

**NSF ECCS 2139713**.